Journal of Sound and Vibration 330 (2011) 510–525

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Generalized thermoelastic extensional and flexural wave motions in homogenous isotropic plate by using asymptotic method

J.N. Sharma, P.K. Sharma*, S.K. Rana

Department of Mathematics, National Institute of Technology, Hamirpur-177005, India

ARTICLE INFO

Article history:
Received 26 February 2010
Received in revised form
3 August 2010
Accepted 8 August 2010
Handling Editor: L.G. Tham
Available online 12 October 2010

ABSTRACT

In this paper the asymptotic method has been applied to investigate propagation of generalized thermoelastic waves in an infinite homogenous isotropic plate. The governing equations for the extensional, transversal and flexural motions are derived from the system of three-dimensional dynamical equations of linear theories of generalized thermoelasticity. The asymptotic operator plate model for extensional and flexural free vibrations in a homogenous thermoelastic plate leads to sixth and fifth degree polynomial secular equations, respectively. These secular equations govern frequency and phase velocity of various possible modes of wave propagation at all wavelengths. The velocity dispersion equations for extensional and flexural wave motion are deduced from the three-dimensional analog of Rayleigh–Lamb frequency equation for thermoelastic plate. The approximation for long and short waves along with expression for group velocity has also been obtained. The Rayleigh–Lamb frequency equations for the considered plate are expanded in power series in order to obtain polynomial frequency and velocity dispersion relations and its equivalence established with that of asymptotic method. The numeric values for phase velocity, group velocity and attenuation coefficients has also been obtained using MATHCAD software and are shown graphically for extensional and flexural waves in generalized theories of thermoelastic plate for solid helium material.

1. Introduction

The theory of thermoelastic waves is well established [1]. The governing field equations in classical dynamic coupled thermoelasticity are wave-type (hyperbolic) equation of motion and diffusion-type (parabolic) equation of heat conduction. It is seen that a part of the solution of energy equation extends to infinity. This means that a part of the disturbance has an infinite velocity of propagation, which is physically unrealistic. The non-classical theories of thermoelasticity have been developed to overcome this drawback. Lord and Shulman [2] incorporated a flux-rate term in Fourier's law of heat conduction in order to formulate a generalized theory that admits finite speed for thermal signals. Green and Lindsay [3] also included a temperature rate term among the constitutive relations to develop a temperature rate dependent thermoelasticity that does not violate the classical Fourier's law of heat conduction when the body under consideration has a center of symmetry. This theory also predicts a finite speed of heat propagation. According to these generalizations, heat propagation should be viewed as a wave phenomenon rather than a diffusion one. A wave-like thermal disturbance is referred to as 'second sound' by Chandrasekharaiah [4]. These theories are also supported by

* Corresponding author. Tel.: +919418047296; +919418479215; +919418340034; fax: 01972223834.
E-mail addresses: jns@nitham.ac.in (J.N. Sharma), psharma@nitham.ac.in (P.K. Sharma), sanjeevrananit@gmail.com (S.K. Rana).

doi:10.1016/j.jsv.2010.08.012

experiments which exhibited the actual occurrence of second sound at low temperatures and small intervals of time. Researchers such as [5–7] experimentally proved for solid helium that thermal waves (second sound) propagating with finite, though quite large, speed also exist. Sharma et al. [8] and Sharma [9] investigated the propagation of thermoelastic Rayleigh–Lamb waves in homogeneous isotropic plates in the context of conventional coupled thermoelasticity (CT) and generalized theories of thermoelasticity under different conditions.

Kirova et al. [10] have studied the asymptotic behavior for linear and nonlinear waves in viscoelastic materials. Ryabenkov and Faizullina [11] proved that asymptotic method is identical with method of hypothesis and successive approximations for slabs and plates. Agalovyan and Gevorkyan [12] solved first boundary-value problem for forced vibrations of an isotropic strip by an asymptotic method. Gales [13] studied asymptotic spatial behavior of solutions in thermoelastic solids. Gevorgyan [14] investigated the thermoelastic wave propagation in a transversely isotropic heat conducting and non-heat conducting elastic materials. Losin [15,16] studied the asymptotic of flexural and extensional waves in homogeneous isotropic elastic plate. Losin [17] established the equivalence of dispersion relations obtained from operator plate model and Rayleigh–Lamb frequency equation. Sharma et al. [18] investigated the flexural and transversal wave motions in homogeneous isotropic thermoelastic plates by using asymptotic method. The authors [15,16] and [18] used the asymptotic method applied by Protsenko [19] for thin n-shelled structures in their investigation on elastic and thermoelastic plates, respectively. Moreover, it is pertinent to mention here that the dispersion relations reported in the works of Losin [15,16] were of sixth degree polynomial equations in frequency/phase velocity instead of tenth degree as reported in [18]. However, the corresponding equivalence relations obtained by Losin [17] in case of symmetric (extensional) and skewsymmetric (flexural) motions of elastic plate are also tenth degree polynomial equations in phase velocity (see terms under the braces of Eqs. (7) and (14) in [17]). Equivalence of these relations has been established by considering terms up to eighth power of $\eta = nh$.

Owing to the technological advances in recent years, plate elements are commonly selected as design components in many engineering structures, especially in the aerospace, marine and construction sectors, because of their ability to resist loads. With the evolution of light plate-structures, tremendous research interests in the vibration of the plates are generated. The negligence of considering vibration as a design factor can lead to excessive deflections and failures. The vibration design aspect is even more important in micro-machines such as electronic packaging, micro-robots, etc. because of their enhanced sensitivities to vibrations. The dynamical problems of the theory of elasticity become increasingly important due to their application in diverse fields. The high velocity of modern aircrafts gives rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure. Keeping in view the above facts and physically realistic nature of non-classical (generalized) thermoelasticity, the present work is an attempt to find a frequency and velocity dispersion relation from three-dimensional analog of the Rayleigh–Lamb frequency equation that would be sufficient for extensional and flexural wave motion in generalized thermoelastic plates. The analysis is based on the approach and asymptotic method of Prosenko [19] used in Refs.[15,16,18] with modification that the approximate matrix inversion by Neumann's series has been replaced by actual matrix inversion. This modification is found very effective as it eliminates restrictions due to the convergence interval for the infinite matrix series and permits the model to be applicable for long and short wave asymptotics in any material [15].

2. BASIC equations and constitutive relations

The constitutive relations and equations governing linear generalized thermoelastic interaction in a homogenous isotropic solid are as follows:

The strain–displacement relations

$$\mathbf{e}_{ij} = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i}), \quad i,j = 1,2,3 \tag{1}$$

The stress–strain temperature relations

$$\boldsymbol{\sigma}_{ij} = \lambda \mathbf{e} \boldsymbol{\delta}_{ij} + 2\mu \mathbf{e}_{ij} - \beta\left(T + t_1 \boldsymbol{\delta}_{2k}\frac{\partial T}{\partial t}\right)\boldsymbol{\delta}_{ij}, \quad i,j = 1,2,3 \tag{2}$$

The equations of motion

$$\frac{\partial \boldsymbol{\sigma}_{ij}}{\partial \mathbf{x}_j} = \rho \frac{\partial^2 \mathbf{u}_i}{\partial t^2}; \quad i,j = 1,2,3 \tag{3}$$

The equation of heat conduction

$$K\nabla^2 T - \rho C_e\left(\frac{\partial T}{\partial t} + t_0\frac{\partial^2 T}{\partial t^2}\right) - \beta T_0\left(\frac{\partial}{\partial t} + t_0 \boldsymbol{\delta}_{1k}\frac{\partial^2}{\partial t^2}\right)\mathbf{e} + \left(1 + t_0\frac{\partial}{\partial t}\right)Q = 0 \tag{4}$$

where $\mathbf{e}_{ij}$ and $\boldsymbol{\sigma}_{ij}$ are the components of strain and stress tensors, respectively; $u=(u_1, u_2, u_3)$ is the displacement vector, $\mathbf{e} = \nabla \cdot \vec{u}$ is the dilatation, ∇ is del operator, ρ is the density, C_e is the specific heat at constant strain. T is the change of temperature from reference temperature T_0; λ, μ are Lame's constant, K is the thermal conductivity, t_0, t_1 are thermal relaxation times, $\boldsymbol{\delta}_{ij}$ and Q are Kronecker delta and heat source term, respectively and β=(3λ+2μ)α_t,α_t being coefficient of linear thermal expansion. $\boldsymbol{\delta}_{ik}$, i=1,2 is Kronecker delta; where k=1 corresponds to Lord–Shulman (LS) and k=2 corresponds to Green–Lindsay (GL) theory of generalized thermoelasticity. According to Strunin [20] the inequalities $t_0 \geq t_1 \geq 0$ of Green [21] obeyed by the thermal relaxation time are not mandatory.

3. Formulation of the problem

We consider wave motion in homogenous isotropic thermoelastic plate of thickness 2h initially at uniform temperature T_0 in the undistributed state. The origin of Cartesian coordinate system '$oxyz$' is taken at any point 'o' in the middle plane of the plate and z-axis is pointing along the thickness of the plate. We assume that the plate is infinite in x and y directions which thus occupies the region

$$\Omega = \{-\infty < x,\ y < \infty,\ -h \leq z \leq h\}$$

In the region Ω, the basic governing equations (3) and (4) in non-dimensional form, in the absence of body forces and heat sources, become

$$\left\{\frac{\partial^2}{\partial x^2} + \delta^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right\}u + (1-\delta^2)\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) - \left(1 + \boldsymbol{\delta}_{2k} t_1 \frac{\partial}{\partial t}\right)\frac{\partial T}{\partial x} = \frac{\partial^2 u}{\partial t^2} \tag{5}$$

$$\left\{\frac{\partial^2}{\partial y^2} + \delta^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right\}v + (1-\delta^2)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial z}\right) - \left(1 + \boldsymbol{\delta}_{2k} t_1 \frac{\partial}{\partial t}\right)\frac{\partial T}{\partial y} = \frac{\partial^2 v}{\partial t^2} \tag{6}$$

$$\left\{\delta^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial z^2}\right\}w + (1-\delta^2)\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z}\right) - \left(1 + \boldsymbol{\delta}_{2k} t_1 \frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} = \frac{\partial^2 w}{\partial t^2} \tag{7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)T - \frac{\partial}{\partial t}\left(1 + t_0 \frac{\partial}{\partial t}\right)T - \varepsilon \frac{\partial}{\partial t}\left(1 + t_0 \boldsymbol{\delta}_{1k} \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{8}$$

where we have used the following non-dimensional quantities

$$\begin{gathered}(x', y', z') = \frac{\omega^*}{c_1}(x, y, z); \quad (u', v', w') = \frac{\rho \omega^* c_1}{\beta T_0}(u, v, w);\\ T' = \frac{T}{T_0}; t' = \omega^* t; \quad t'_1 = \omega^* t_1; \quad t'_0 = \omega^* t_0; \quad h' = \frac{\omega^* h}{c_1}; \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}; \quad c_2^2 = \frac{\mu}{\rho};\\ \delta^2 = \frac{c_2^2}{c_1^2} = \frac{\mu}{\lambda + 2\mu}; \varepsilon = \frac{\beta^2 T_0}{\rho c_e(\lambda + 2\mu)}; \quad \omega^* = \frac{(\lambda + 2\mu)c_e}{K}; \quad \boldsymbol{\sigma}'_{ij} = \frac{\boldsymbol{\sigma}_{ij}}{\beta T_0}\end{gathered} \tag{9}$$

Here ε is thermoelastic-coupling constant, u, v, and w are displacement components, and c_1, c_2 are the velocities of longitudinal and transverse waves, respectively. In Eqs. (5)–(8) and in the following analysis, the primes have been suppressed for convenience unless stated otherwise.

The surfaces $z = \pm h$ of the plate are assumed to be stress free and thermally insulated. Therefore, the non-dimensional boundary conditions to be satisfied are given as

$$\begin{gathered}\delta^2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0\\ \delta^2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0\\ (1-2\delta^2)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial z} - \left(1 + \boldsymbol{\delta}_{2k} t_1 \frac{\partial}{\partial t}\right)T = 0\\ \frac{\partial T}{\partial z} = 0\end{gathered} \tag{10}$$

4. Solution of the problem by asymptotic method

We assume harmonic wave solution of the form

$$(u, T, v, w)(x, y, z, t) = \vec{\mathbf{u}}(z)\exp\{-i(\vec{\mathbf{r}} \cdot \vec{\mathbf{n}} - \omega t)\} \tag{11}$$

where $\vec{\mathbf{u}}(z) = (U(z),\ \theta(z),\ V(z),\ W(z))$ is amplitude vector, ω is the circular frequency depending on the wavenumber $\vec{\mathbf{n}} = (n_1,\ n_2)$ and position vector $\vec{\mathbf{r}} = (x,\ y)$.

Using solution (11) in governing equations (5)–(9) and boundary conditions (10), we obtain

$$\left(\frac{d^2}{dz^2}-\hat{\mathbf{Q}}_1\frac{d}{dz}-\hat{\mathbf{R}}_1\right)\vec{\mathbf{u}}(z)=\vec{\mathbf{0}},\ \text{in the domain } \Omega \tag{12}$$

$$\boldsymbol{\sigma}(z,n)=\left(\frac{d}{dz}-\hat{\mathbf{S}}_1\right)\vec{\mathbf{u}}(z)=\vec{\mathbf{0}},\ \text{on } z=\pm h \tag{13}$$

where $\hat{\mathbf{Q}}_1=n\mathbf{Q}_1,\ \hat{\mathbf{R}}_1=n^2\mathbf{R}_1,\ \hat{\mathbf{S}}_1=n\mathbf{S}_1$

$$\mathbf{Q}_1=(q_{ij})_{4\times4},\ \mathbf{R}_1=(r_{ij})_{4\times4},\ \mathbf{S}_1=(s_{ij})_{4\times4} \tag{14}$$

are 4×4 order matrices and $\boldsymbol{\sigma}(z,n)=\begin{bmatrix}\delta^{-2}\sigma_{xz} & T_{,z} & \delta^{-2}\sigma_{yz} & \sigma_{zz}\end{bmatrix}^t$ is thermal stress vector.

Here the non-zero elements of matrices $\mathbf{Q}_1$, $\mathbf{R}_1$ and $\mathbf{S}_1$ are given as

$$q_{14}=i\overline{n}_1(\delta^{-2}-1),\quad q_{24}=-\varepsilon c\omega\tau'_0,\quad q_{34}=i\overline{n}_2(\delta^{-2}-1),\quad q_{41}=i\overline{n}_1(1-\delta^2),\quad q_{42}=-ic\tau_1,$$
$$q_{43}=i\overline{n}_2(1-\delta^2),\quad r_{11}=\delta^{-2}\overline{n}_1^2+\overline{n}_2^2-\delta^{-2}c^2,\quad r_{12}=-\tau_1\overline{n}_1c\delta^{-2},\quad r_{13}=\overline{n}_1\overline{n}_2(\delta^{-2}-1),$$
$$r_{21}=-i\omega\overline{n}_1\varepsilon c\tau'_0,\quad r_{22}=\overline{n}_1^2+\overline{n}_2^2+\tau_0c^2,\quad r_{23}=-i\varepsilon\overline{n}_2c\omega\tau'_0,\quad r_{31}=(\delta^{-2}-1)\overline{n}_1\overline{n}_2,\quad r_{32}=-\tau_1\overline{n}_2c\delta^{-2},$$
$$r_{33}=\overline{n}_1^2+\delta^{-2}\overline{n}_2^2-\delta^{-2}c^2,\quad r_{44}=\delta^2(\overline{n}_1^2+\overline{n}_2^2)-c^2,\quad s_{14}=i\overline{n}_1,\quad s_{34}=i\overline{n}_2,\quad s_{41}=i(1-2\delta^2)\overline{n}_1,$$
$$s_{42}=-i\tau_1c,\quad s_{43}=i(1-2\delta^2)\overline{n}_2$$

where

$$\tau'_0=i\omega^{-1}-t_0\delta_{1k};\quad \tau_1=i\omega^{-1}-t_1\delta_{2k},\quad \tau_0=i\omega^{-1}-t_0,\quad \overline{n}_1=n_1/n,\quad \overline{n}_2=n_2/n,\quad n=|\vec{\mathbf{n}}|=\sqrt{n_1^2+n_2^2},\quad \vec{\mathbf{n}}=n\,\hat{\mathbf{n}},\quad \hat{\mathbf{n}}=(\overline{n}_1,\ \overline{n}_2),$$

$\vec{\mathbf{c}}(\vec{\mathbf{n}})=\vec{\boldsymbol{\omega}}(n)/n$ is the phase velocity and c is the phase speed of a traveling wave and $\mathbf{n}$ is the unit direction vector.

For waves propagating along x-axis, $(\overline{n}_1,\ \overline{n}_2)=(1,\ 0)$ and hence Eqs. (12) and (13) become

$$\left(\frac{d^2}{dz^2}-\mathbf{Q}^*\frac{d}{dz}-\mathbf{R}^*\right)\vec{\mathbf{u}}(z)=\vec{\mathbf{0}},\ \text{in the domain } \Omega$$

$$\boldsymbol{\sigma}(z,n)=\left(\frac{d}{dz}-\mathbf{S}\right)\vec{\mathbf{u}}(z)=\vec{\mathbf{0}},\ \text{on } z=\pm h$$

where the matrices $\mathbf{Q}^*$, $\mathbf{R}^*$ and $\mathbf{S}$ can be obtained from $\mathbf{Q}_1$, $\mathbf{R}_1$ and $\mathbf{S}_1$ defined in Eq. (14) by setting$(\overline{n}_1,\ \overline{n}_2)=(1,\ 0)$.

Employing finite asymptotic expansions [18] for $\boldsymbol{\sigma}(z,n)$ and eliminating higher order derivatives, we obtain

$$\left[\mathbf{I}+\frac{h^2n^2}{2}\mathbf{A}+\frac{h^4n^4}{24}\mathbf{B}\right]\vec{\mathbf{u}}^{(1)}(0)-n\left[\mathbf{S}-\frac{h^2n^2}{2}\mathbf{C}-\frac{h^4n^4}{24}\mathbf{N}\right]\vec{\mathbf{u}}(0)=\vec{\mathbf{0}} \tag{15}$$

$$\left[\mathbf{Q}^*-\mathbf{S}+\frac{h^2n^2}{6}\mathbf{E}+\frac{h^4n^4}{120}\mathbf{F}\right]\vec{\mathbf{u}}^{(1)}(0)-n\left[\mathbf{R}^*-\frac{h^2n^2}{6}\mathbf{G}-\frac{h^4n^4}{120}\mathbf{H}\right]\vec{\mathbf{u}}(0)=\vec{\mathbf{0}} \tag{16}$$

where $\vec{\mathbf{u}}^{(1)}(0)=\left(\frac{d\vec{\mathbf{u}}}{dz}\right)_{z=0}$

$$\mathbf{A}=\mathbf{R}^*+(\mathbf{Q}^*-\mathbf{S})\mathbf{Q}^*,\quad \mathbf{B}=\mathbf{G}+\mathbf{E}\mathbf{Q}^*,\quad \mathbf{C}=(\mathbf{Q}^*-\mathbf{S})\mathbf{R}^*,$$
$$\mathbf{E}=\mathbf{C}+\mathbf{A}\mathbf{Q}^*,\quad \mathbf{F}=\mathbf{B}\mathbf{Q}^*+\mathbf{N},\quad \mathbf{G}=\mathbf{A}\mathbf{R}^*,\quad \mathbf{H}=\mathbf{B}\mathbf{R}^*,\quad \mathbf{N}=\mathbf{E}\mathbf{R}^*$$

It is noticed that the matrices $\mathbf{A}$ and $\mathbf{B}$ have block diagonal structures of the type

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1^{(1)} & \mathbf{O}\\ \mathbf{O} & \mathbf{A}_2\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}\mathbf{B}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{B}_2\end{bmatrix}$$

where$\mathbf{A}_1=(\mathbf{a}_{ij})_{2\times2}$, $\mathbf{B}_1=(\mathbf{b}_{ij})_{2\times2}$, $\mathbf{A}_2=\mathbf{diag}(a_{33},\ a_{44})$, $\mathbf{B}_2=\mathbf{diag}(b_{33}, b_{44})$.

Here the elements of these matrices are given below:

$$a_{11}=3-2\delta^2-v_s^2,\quad a_{12}=-2\delta\tau_1v_s,\quad a_{21}=\varepsilon_1\tau_1^{-1}\delta(\delta^2-2)v_s,\quad a_{22}=1+\delta^2(\tau_0+\varepsilon_1\tau_1^{-1}\varepsilon^{-1})v_s^2,$$
$$a_{33}=1-v_s^2,\quad a_{44}=-1+2\delta^2-\delta^2v_s^2,\quad b_{11}=5-4\delta^2-m_1v_s^2+v_s^4,$$
$$b_{21}=\varepsilon_1\tau_1^{-1}\delta v_s\left[2(2\delta^2-3)-m_2v_s^2\right],\quad b_{12}=-4\delta\tau_1v_s-2\tau_1\delta m_3v_s^3,\quad b_{22}=1+m_4v_s^2+m_5v_s^4,$$
$$b_{33}=(1-v_s^2)^2,\quad b_{44}=4\delta^2-3-m_6v_s^2+m_7v_s^4,\quad v_s=\frac{c}{\delta},\quad m_1=2\delta^2(2\delta^{-2}-1)(1-\xi\tau_1\delta^2),$$
$$m_2=-2+\delta^2[2\tau_0+\varepsilon_1(2-\delta^2)+\delta^2(1-\tau_0)],\quad m_3=\delta^2[(\tau_0-1)+\varepsilon_1],\quad m_4=2\delta^2(\tau_0+2\varepsilon_1),$$
$$m_5=\delta^4[\tau_0{}^2+\varepsilon_1(\tau_0-1)],\quad m_6=2[-1+\delta^2(\overline{\delta}_1-\varepsilon_1\delta^2)],\quad m_7=\delta^4(1-\varepsilon_1),\quad \delta_1=1-\delta^2,\quad \overline{\delta}_1=1+\delta^2,\quad \varepsilon_1=i\omega\tau'_0\tau_1\varepsilon$$

Noting that the coefficient of $\vec{\mathbf{u}}^{(1)}(0)$ in Eq. (15) is a non-singular square matrix of order four, one can obtain the resolving operator from Eqs. (15) and (16) as

$$\mathbf{P}\vec{\mathbf{u}}(0)=\left(\mathbf{P}_0+\mathbf{P}_2\frac{h^2n^2}{6}+\mathbf{P}_4\frac{h^4n^4}{120}\right)\vec{\mathbf{u}}(0)=\vec{\mathbf{0}} \tag{17}$$

where

$$\mathbf{P}_0=\mathbf{R}^*+(\mathbf{Q}^*-\mathbf{S})\mathbf{M}^{-1}\mathbf{S},$$
$$\mathbf{P}_2=\mathbf{G}+\mathbf{E}\mathbf{M}^{-1}\mathbf{S}-3(\mathbf{Q}^*-\mathbf{S})\mathbf{M}^{-1}\mathbf{C},$$
$$\mathbf{P}_4=\mathbf{H}+\mathbf{F}\mathbf{M}^{-1}\mathbf{S}-10\mathbf{E}\mathbf{M}^{-1}\mathbf{C}-5(\mathbf{Q}^*-\mathbf{S})\mathbf{M}^{-1}\mathbf{N}$$

Here the matrix$\mathbf{M}=(\mathbf{m}_{ij})_{4\times 4}$; $\mathbf{m}_{ij}=\boldsymbol{\delta}_{ij}+(h^2n^2/2)\mathbf{a}_{ij}+(h^4n^4/24)\mathbf{b}_{ij}$, $(i,j=1,2,3,4)$ and $\mathbf{M}^{-1}$ is its inverse. The matrix of the operator $\mathbf{P}$ has in general a block diagonal structure of the form

$$\mathbf{P}=\mathbf{diag}(\mathbf{P}_L,\mathbf{P}_{S_1},\mathbf{P}_{S_2}) \tag{18}$$

thus we have

$$\mathbf{P}_L\begin{bmatrix}U\\ \theta\end{bmatrix}=0,\quad \mathbf{P}_{S_1}[V]=0,\quad \mathbf{P}_{S_2}[W]=0$$

where $\mathbf{P}_L=(\mathbf{p}_{ij})_{2\times 2}$, $\mathbf{P}_{S_1}=(p_{33})_{1\times 1}$ and $\mathbf{P}_{S_2}=(p_{44})_{1\times 1}$, respectively, govern the extensional, transversal and flexural in plane motion of the plate. Eq. (17) has a non-trivial solution if and only if determinant

$$|\mathbf{P}|=0 \tag{19}$$

This leads to the secular equations

$$p_{11}p_{22}-p_{12}p_{21}=0,\quad p_{33}=0,\quad p_{44}=0 \tag{20}$$

Eqs. (20) are the three-dimensional analog of the Rayleigh–Lamb frequency equations for extensional, transversal and flexural wave motion of a thermoelastic plate. The second equation in system (20) corresponds to the frequency equation for transversal wave motion which remains independent of thermal variations and has already been discussed in Sharma et al. [18]. In the following we confine our discussion to the study of extensional and flexural wave motions in the generalized thermoelastic plate.

5. Extensional motion of a plate

Because the operator $\mathbf{P}_L$ affects the displacement U and temperature θ only, hence first equation of system (20) governs the extensional vibrations. According to the structure (17) the operator $\mathbf{P}$, the first equation in the system of equations (20) provides us the extensional wave phase velocity equation as

$$e_0v_s^{12}+h_1v_s^{10}+h_3v_s^8+h_6v_s^6+h_9v_s^4+h_{12}v_s^2+h_{15}=0 \tag{21}$$

where $h_1=e_1+20e_2/n^2h^2$, $\mathbf{h}_j=\mathbf{e}_j+20\mathbf{e}_{j+1}/n^2h^2+120\mathbf{e}_{j+2}/n^4h^4 j=3i,\ i=1,2,3,4,5$

The quantities e_j (j=0 to 17) are defined as

$$e_0=-a_3q_5,\ e_1=a_3q_4-a_2q_5+\delta\tau_1b_3d_3,\ e_2=-a_3q_2+(1+3\delta^2)q_5,$$
$$e_3=a_2q_4+a_3q_3+\delta_3q_5-\delta\tau_1(b_3d_2-b_2d_3),$$
$$e_4=a_3q_1+a_1q_5-a_2q_2-(3\delta^2+1)q_4-\delta\tau_1(b_1d_3+b_3d_1),$$
$$e_5=\frac{40}{3}(3\delta^2+1)q_2-(q_5+a_3\gamma_1),$$
$$e_6=a_2q_3+32\delta_1q_5-a_3\delta_5-\delta_3q_4-12\varepsilon_1\delta_1\delta^2d_3-\delta\tau_1(b_2d_2-16b_3),$$
$$e_7=a_2q_1-a_1q_4-a_3\delta_4+\delta_3q_2-\delta_2q_3+8\delta_1q_5+2\delta_1\delta^2\varepsilon_1d_3+\delta\tau_1(b_2d_1-b_1d_2+4b_3),$$
$$e_8=q_4-a_2\gamma_1+4q_5\delta_1-a_3+\frac{10}{3}(a_1q_2-q_1\delta_2)+2\varepsilon_1\delta_1\delta^2d_3+\delta\tau_1\left(2b_3+\frac{10}{3}b_1d_1\right),$$
$$e_9=-\left[q_3\delta_3+32q_4\delta_1+a_2\delta_5+12\varepsilon_1\delta_1\delta^2d_2+16\delta\tau_12b_2\right],$$
$$e_{10}=-\left[q_3a_1+a_2\delta_4-32q_2\delta_1+q_1\delta_3+8q_4\delta_1-\delta_5\delta_2+2\varepsilon_1\delta_1\delta^2(6d_1-d_2)+4\delta\tau_1(4b_1+b_2)\right],$$
$$e_{11}=-\left[a_2-q_3+4q_4\delta_1-\gamma_1\delta_3+\frac{10}{3}(q_1a_1-8q_2\delta_1-\delta_4\delta_2)+2\varepsilon_1\delta_1\delta^2\left(\frac{10}{3}d_1-d_2\right)+2\delta\tau_1\left(b_2+\frac{20}{3}b_1\right)\right],$$
$$e_{12}=\delta_5\delta_3-32q_3\delta_1-192\varepsilon_1\delta_1\delta^2,\ e_{13}=a_1\delta_5-32q_1\delta_1-8q_3\delta_1+\delta_4\delta_3-16\varepsilon_2\delta_1\delta^2,$$
$$e_{14}=32\gamma_1\delta_1-4q_3\delta_1+\delta_3-\delta_5+\frac{10}{3}(a_1\delta_4-8q_1\delta_1)+\frac{173}{5}\varepsilon_1\delta_1\delta^2,\ e_{15}=32\delta_1\delta_5,$$
$$e_{16}=-256\delta_1(2\delta^2-1),\ e_{17}=\frac{53}{5}\delta_1(30\delta^2-11)$$

where $\delta_2=1+3\delta^2$, $\delta_3=16(2\delta^4-3)$,$\delta_4=6\delta^2-2$,$\delta_5=8(5\delta^2-3)$

and $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{d}_i$ (i=1,2,3), $\mathbf{q}_j$ (j=1, …, 5) are given in Eqs. (A.1–A.8).

Adopting the procedure of Sharma et al. [18], the frequency equation and group velocity in the present case are obtained as

$$e_0\omega_s^{12}+h_1n^2\omega_s^{10}+h_3n^4\omega_s^8+h_6n^6\omega_s^6+h_9n^8\omega_s^4+h_{12}n^{10}\omega_s^2+h_{15}n^{12}=0 \tag{22}$$

$$c_g=-\frac{1}{v_s}\left(\frac{e_1v_s^{10}+h_1^*v_s^8+(h_6+h_2^*)v_s^6+(2h_9+h_3^*)v_s^4+(3h_{12}+h_4^*)v_s^2+5h_6+h_5^*}{6e_0v_s^{10}+5h_1v_s^8+4h_3v_s^6+3h_6v_s^4+2h_9v_s^2+h_{12}}\right) \tag{23}$$

where $\mathbf{h}_i^*=2\mathbf{e}_j+\mathbf{e}_{j+1}/n^2h^2$, j=3i, i=1,2,3,4,5.

5.1. Long and short wavelength waves

In case of long wavelength ($nh\to 0$) and short wavelength ($nh\to\infty$) limits, Eq. (21) reduces to

$$\omega_s^8\left(e_0\omega_s^4+\frac{e_2}{h^2}\omega_s^2+\frac{e_5}{h^4}\right)=0 \tag{24}$$

$$e_0v_s^{12}+e_1v_s^{10}+e_3v_s^8+e_6v_s^6+e_9v_s^4+e_{12}v_s^2+e_{15}=0 \tag{25}$$

respectively. Eq. (24) clearly has one trivial root of multiplicity four and corresponding phase velocity is $v_s=2\sqrt{1-\delta^2/(1+\varepsilon)}$. The six pairs of roots of Eq. (25) are phase velocities of incoming and outgoing short wavelength modes which depend on the parameters(ε, δ) only. Because the roots are, in general complex, therefore waves will be attenuating in space and dispersive in character.

6. Flexural motion of a plate

Following the procedure of Sharma et al. [18], the phase velocity and group velocity equations, are obtained as

$$f_0v_s^{10}+g_1v_s^8+g_3v_s^6+g_6v_s^4+g_9v_s^2+g_{12}=0 \tag{26}$$

$$c_g=-\frac{1}{v_s}\left(\frac{f_1v_s^8+g_1^*v_s^6+(g_3+g_2^*)v_s^4+(2g_4+g_3^*)v_s^2+(3g_5+g_4^*)}{5f_0v_s^8+4g_1v_s^6+3g_2v_s^4+2g_3v_s^2+2g_4}\right) \tag{27}$$

where $g_1=f_1+f_2/n^2h^2$, $\mathbf{g}_j=\mathbf{f}_j+\mathbf{f}_{j+1}/n^2h^2+\mathbf{f}_{j+2}/n^4h^4$,$\mathbf{g}_i^*=\mathbf{2f}_j+\mathbf{f}_{j+1}/n^2h^2$, j=3i, i=1,2,3,4.

Here the quantities f_j (j=0 to 14) are obtained as

$$\begin{gathered}
f_0=-120\delta^2(m_7p_3+240\varepsilon_1\varepsilon^{-1}\delta^4),\\
f_1=120\delta^2[m_7(p_2+p_3)+m_6p_3]+2880[m_5p_5-10m_3p_7-\tau_1^{-1}\varepsilon_1\delta\{m_2p_9+10\tau_1\delta^5(m_1-3)\}],\\
f_2=6\delta^2(20p_3\delta^2-m_7p_1+2880\varepsilon_2\delta^4),\\
f_3=-120\delta^2[B_1+\varepsilon_1\tau_1^{-1}\delta\{m_2p_8+2p_9(2\delta^2-3)+10\tau_1\delta^5(3m_1+4\delta^2-7)\}],\\
f_4=6\delta^2B_2-576[3(\gamma_4p_5+\gamma_3p_9-10\delta\tau_1p_7)-5\varepsilon_1\delta^4m_2+\delta\, m_3+60\delta^2(\gamma_5+3)\}],\\
f_5=4\delta^2[144m_7-5p_1\delta^2+30(m_3+40\delta^4\varepsilon_1)],\\
f_6=120\delta^2[B_3+m_6(p_2+\beta_1)-240\varepsilon_1\tau_1^{-1}\delta^{-1}\{m_2p_{10}+2p_8(2\delta^2-3)+10\delta^5\tau_1(4\delta^2-7+3m_1)\}],\\
f_7=-6\delta^2B_4+1728[p_5-p_4\gamma_4-p_8\gamma_3+10p_5\delta\tau_1]+5760\varepsilon_1\delta^4[m_2(1+3\delta^2)+\delta^2(2+9\beta_5)-3],\\
f_8=120B_5+576[p_5+(m_6+m_7)\delta^2],\\
f_9=120B_6-240\varepsilon_1\tau_1^{-1}\delta[2\delta^2-3+\delta^5\tau_1(4\delta^2-5)],\\
f_{10}=-120\delta^2[B_7-1728\{p_4-\delta^2(\tau_0+\varepsilon_1)\gamma_7+10p_0\delta\tau_1\}-240\varepsilon_1\delta\{-5+6\delta^3\gamma_5+2\delta(2\delta^2-3)\}],\\
f_{11}=576\delta^2[189-16\delta^2(5\tau_0\delta_1+\delta^2+7+\varepsilon_1\delta_1)],\\
f_{12}=-23040\delta^2(3\delta^4-6\delta^2+4),\quad f_{13}=-1152\delta^2(432\delta^4-839\delta^2+318)\\
f_{14}=64512\delta^2\delta_1
\end{gathered}$$

The quantities $\mathbf{B}_j$ (j=1, …, 7) and $\mathbf{p}_i$ (i=1, …, 10) are defined by Eqs. (A.9)–(A.21).

6.1. Long and short wavelength flexural waves

In case of long wavelength ($nh\to 0$) and short wavelength ($nh\to\infty$) limits, Eq. (26) reduces to

$$\omega_s^6\left(f_0\omega_s^4+\frac{f_2}{h^2}\omega_s^2+\frac{f_4}{h^4}\right)=0 \tag{28}$$

$$f_0v_s^{10}+f_1v_s^8+f_3v_s^6+f_6v_s^4+f_9v_s^2+f_{12}=0 \tag{29}$$

respectively. Clearly Eq. (28) has one trivial root (ω_s^2=0) of multiplicity three with corresponding phase velocity equal to zero and two pairs non-trivial roots. The solution of Eq. (29) gives the phase velocities (v_s) of five pairs of incoming and outgoing wave modes (in general complex) as function parameters(ε, δ). Hence, the wave motion under consideration in this case is dispersive and attenuating in character.

7. Equivalence with thermoelastic Rayleigh–Lamb wave equation

Consider the thermoelastic Rayleigh–Lamb frequency equations for a thermoelastic plate [8], in the context of generalized theories of thermoelasticity

$$\left[\frac{\tan m_1 h}{\tan \beta_1 h}\right]^{\pm 1} - \frac{m_1(\alpha^2 - m_1^2)}{m_3(\alpha^2 - m_3^2)}\left[\frac{\tan m_3 h}{\tan \beta_1 h}\right]^{\pm 1} = \frac{4\beta_1 m_1 n^2 (m_3^2 - m_1^2)}{(n^2 - \beta_1^2)^2(\alpha^2 - m_3^2)} \tag{30}$$

where $\alpha^2 = n^2(c^2 - 1)$, $\beta_1^2 = n^2(c^2/\delta^2 - 1)$, $m_1^2 = n^2(a^2c^2 - 1)$, $m_3^2 = n^2(b^2c^2 - 1)$, $a^2,\ b^2 = \{(1+\tau_0 - i\omega\varepsilon\tau'_0\tau_1) \pm \sqrt{(1-\tau_0 - i\omega\varepsilon\tau'_0\tau_1)^2 - 4i\omega\varepsilon\tau_0\tau'_0\tau_1}\}/2$.

Here, the positive and negative powers correspond to the flexural and extensional wave modes, respectively. After expanding all tangent terms into the power series (considering only negative exponent), Eq. (30) takes the form:

$$\psi_0 + \frac{1}{3}h^2\psi_2 + \frac{2}{15}h^4\psi_4 + \frac{2}{45}h^6\psi_6 + \frac{4}{225}h^8\psi_8 + \cdots = 0 \tag{31}$$

where

$$\begin{gathered}
\psi_0 = \phi_0(n^2 - \beta_1^2)^2 - 4n^2 m_1^2 m_3^2 \\
\psi_2 = (\phi_2 + \beta_1^2\phi_0)(n^2 - \beta_1^2)^2 - 4n^2 m_1^2 m_3^2 \theta_2 \\
\psi_4 = [2(\phi_4 + \beta_1^4\phi_0) + 5/3\beta_1^2\phi_2](n^2 - \beta_1^2)^2 - 4n^2 m_1^2 m_3^2(\theta_4 + 5/3 m_1^2 m_3^2) \\
\psi_6 = {\beta_1}^2[(\theta_4 + m_1^2 m_3^2)(\alpha^2 - \beta_1^2) + \theta_2(\beta_1^2\alpha^2 - \theta_4)](n^2 - \beta_1^2)^2 - 4n^2 m_1^4 m_3^4 \theta_2 \\
\psi_8 = \beta_1^4[\alpha^2(\theta_4 + m_1^2 m_3^2) + \theta_2\theta_4](n^2 - \beta_1^2)^2 - 4n^2 m_1^6 m_3^6, \\
\theta_i = (m_1^i + m_3^i), \quad \phi_0 = \alpha^2 - \theta_2, \quad \phi_2 = \theta_2\alpha^2 - (\theta_4 + m_1^2 m_3^2), \\
\phi_4 = \alpha^2(\theta_4 + m_1^2 m_3^2) - \theta_2\theta_4
\end{gathered}$$

It can be established that all the coefficients of $\psi_n(c,\delta)$ are completely identical to those of $\mathbf{P}_L$ in operator $\mathbf{P}$ of Eq. (18) associated with the extensional motion in the operator plate model. That is we have

$$\psi_n(c,\ \delta) \cong {P_L}^{(n)}(c,\delta), \quad (n = 0,\ 2,\ 4,\ 6,\ \ldots) \tag{32}$$

This type of motion is governed by the dispersion relation $P_L(c,\delta)=0$, reproduced here with the corresponding order of approximation

$$\nu_s^2\left(P_L^{(0)}(c,\ \delta) + \frac{1}{3}\eta^2 P_L^{(2)}(c,\ \delta) + \frac{2}{15}\eta^4 P_L^{(4)}(c,\ \delta) + \frac{17}{315}\eta^6 {P_L}^{(6)}(c,\delta) + \frac{62}{2835}\eta^8 P_L^{(8)}(c,\ \delta) + \cdots\right) = 0 \tag{33}$$

where $\eta=hn$

$$\begin{gathered}
P_L^{(0)}(c,\delta) = (\delta^2 - \varphi_2){\nu_s}^4 + [1 + 4(\varphi_2 - \overline{\varphi}_2 - \delta^2]\nu_s^2 - 4\delta_1, \\
P_L^{(2)}(c,\delta) = -(\varphi_4 + \varphi_2\delta_1 + \overline{\varphi}_2 - \delta^2)\nu_s^6 + [1 - 7\delta^2 + 4\overline{\varphi}_2\delta^{-2} + 4\varphi_4 + \varphi_2(3 + 4\delta_1 - 4\overline{\varphi}_2\delta^{-2})]\nu_s^4, \\
-4[1.5 - 4\delta^2 - \overline{\varphi}_2\delta^{-2} + \varphi_4 + \varphi_2(3 + \delta_1 - \varphi_2)]\nu_s^2 + 12\delta_1, \\
P_L^{(4)}(c,\delta) = -0.3[\varphi_2(6\varphi_4 - 5\delta^2) + (\varphi_4 + \overline{\varphi}_2)(5 - 6\delta^2)]\nu_s^8 + 0.3r_1\nu_s^6 - 0.3r_2\nu_s^4, \\
-0.3[4\varphi_2 + (7\varphi_2 + 23\delta^2 - 38) - 40\varphi_4 + 4\overline{\varphi}_2(17 + 38\delta^2) - 31]\nu_s^2 + 2.4(25\varphi_2 + 14\delta^2 - 8), \\
P_L^{(6)}(c,\delta) = -[(\varphi_4 + \overline{\varphi}_4)\delta_1 - \varphi_2(\varphi_4 - \delta^2)]\nu_s^{10} + r_6\nu_s^8 + r_5\nu_s^6 + r_4\nu_s^4, \\
+2[24\delta^2 - 9 - 16\overline{\varphi}_2 - 4\varphi_2(5 + \delta_1)]\nu_s^2 + 20\delta_1, \\
P_L^{(8)}(c,\delta) = [(\overline{\varphi}_2 + \varphi_4)\delta^2 - \varphi_2\varphi_4]\nu_s^{12} - r_{10}{\nu_s}^{10} + r_9\nu_s^8 - r_8\nu_s^6 + r_7\nu_s^4, \\
-[36\delta^2 - 13 + 10\varphi_4 + 4\varphi_2(3\delta^2 - 9 - 2\varphi_2)]\nu_s^2 - 12\delta_1)
\end{gathered} \tag{34}$$

Here the quantities $\mathbf{r}_i$ (i=1 to 10) and $\overline{\varphi}_i$ ($i = 2,4$) are defined by Eqs. (A.22–A.32).

The only difference between Eqs. (21) and (33) is in factors outside the braces. Eq. (33) has an extra trivial solution and the identity of expressions inside the braces guarantees the equivalence of non-trivial roots for extensional modes. Similarly after expanding all the tangent functions into the power series (considering only positive exponent), Eq. (30) takes the form as that of Eq. (42) in Sharma et al. [18]. The resulting equation also has an extra trivial solution and the identity of expressions inside the braces guarantees the equivalence of non-trivial roots in case of flexural modes.

8. Special cases

In this section the wave motion of elastic plate under isothermal and isentropic (adiabatic) conditions has been discussed.

(a) *Elastic plate under thermal equilibrium*

In the case of uncoupled thermoelasticity (elastic plate), the coefficient of linear thermal expansion vanishes because the elastic and thermal fields are independent of each other so that β=0 which implies that ε=0. The dispersion equation (21) for extensional motion reduced to tenth degree polynomial in υ_s whereas the secular equation (26) for flexural motion preserves its nature with same degree. These equations provide us five modes of wave propagation each for extensional and flexural motions of a homogeneous isotropic elastic plate. This is in agreement with Sharma and Kumar [24]. The Rayleigh–Lamb frequency equation (30) in this case gets reduced to the dispersion relation of elastic plate waves [22,26] for ε=0 and the consequent equivalence of results agree with Losin [17].

(b) *Elastic plate under isentropic conditions*

In this case of thermoelasticity, the coefficient of thermal conductivity vanishes, so that K=0. Consequently the dispersion equation (21) for extensional motion gets reduced to tenth degree polynomial in υ_s here in contrast to the flexural motion given by polynomial equation (26) which preserve its degree and remains similar in nature. Thus there exist which give five modes of wave propagation for each extensional and flexural motions in a homogeneous isotropic elastic plate.

9. Numerical results and discussion

For the purpose of numerical illustrations, we consider wave propagation in an infinite, homogenous isotropic, thermoelastic plate of solid helium material. The physical data of solid helium crystal as reported in Sharma and Kumar [23] given by

$$\lambda = 0.2120 \times 10^{10}\ \mathrm{N/m^2}, \quad \mu = 0.1245 \times 10^{10}\ \mathrm{N/m^2}, \quad \varepsilon = 0.04162,$$
$$\rho = 0.1910\ \mathrm{kg/m^3}, \quad \beta = 2.3620 \times 10^{6}\ \mathrm{N/m^2\,deg^{-1}}, \quad K = 0.3000 \times 10^{2}\ \mathrm{W\,m^{-1}\,deg^{-1}}$$
$$\omega^* = 1.9890 \times 10^{13}\ \mathrm{s^{-1}}$$

In general, the secular equations are complex polynomial equations and hence provide us complex phase velocities at first instant. All the modes are found to be dispersive and attenuated in character. The computations have been carried out to compute phase and group velocities and attenuation coefficients of various modes of wave propagation in case of CT, LS and GL theories of thermoelasticity correct upon four decimal places here.

If we write

$$c^{-1} = V^{-1} + i\omega^{-1}Q \tag{35}$$

so that $n=R+iQ$, where$R=\omega/V$, V, Q, ω are real numbers.

The real phase speeds (V_i) and attenuation coefficients(Q_i) can be obtained from the complex phase velocity (c_i) by using representation (35). The Characteristic equations (21) and (26) have been solved by using Graeffe's root squaring method [25] with the help of MATHCAD software. This method has two major advantages over the other methods namely: firstly it requires no prior information about the roots of an equation and secondly it is capable of giving all the roots at a time. The various computed quantities have been normalized with their corresponding values in case of coupled thermoelasticity (CT).

The variations of phase(V_i, $i=1,2,3,4,5$) and group(V_{g_i}, $i=1,2,3,4,5$) velocities of extensional modes in elastic (isothermal and isentropic) plate with wavenumber (Rh) are presented in Figs. 1 and 2, respectively. The variations of phase and group velocities of different modes of propagation in thermoelastic plate are presented in Figs. 3 and 4. The variations of phase and group velocities of extensional modes of (isothermal and isentropic) elastic plate are similar to that of thermoelastic plate as observed from Figs. 1–4 with the exception that the magnitudes of phase velocities of various wave modes in coupled thermoelastic plate are small as compared to that in elastic plate (isothermal or isentropic) at all wavelengths. The phase velocity (V_1) of fundamental mode of extensional motion decreases from a value greater than

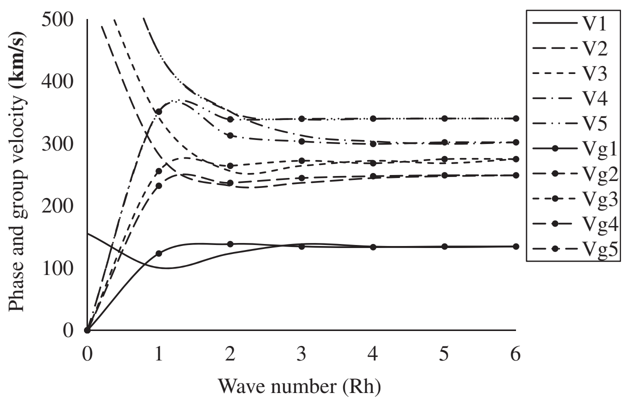

Fig. 1. Phase and group velocities of extensional modes in isothermal elastic plate versus wavenumber (Rh).

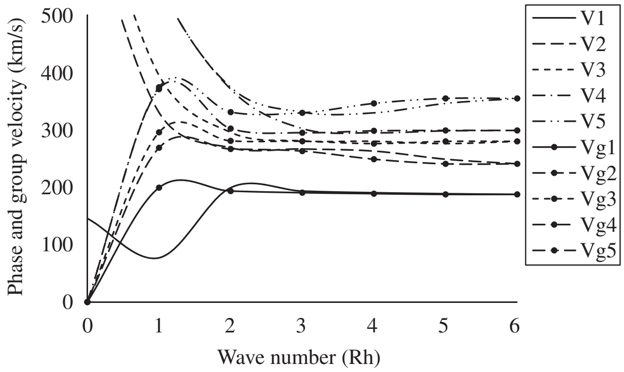

Fig. 2. Phase and group velocities of extensional modes in non-conducting (isentropic) elastic plate versus wavenumber (*Rh*).

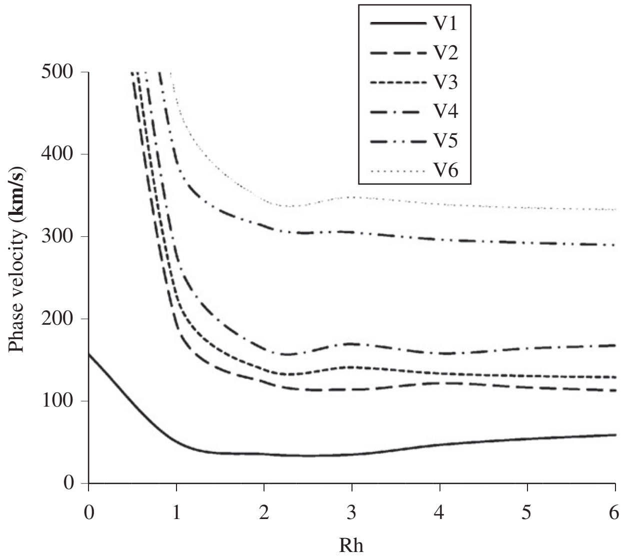

Fig. 3. Phase velocity of extensional modes versus wavenumber (*Rh*) for CT theory.

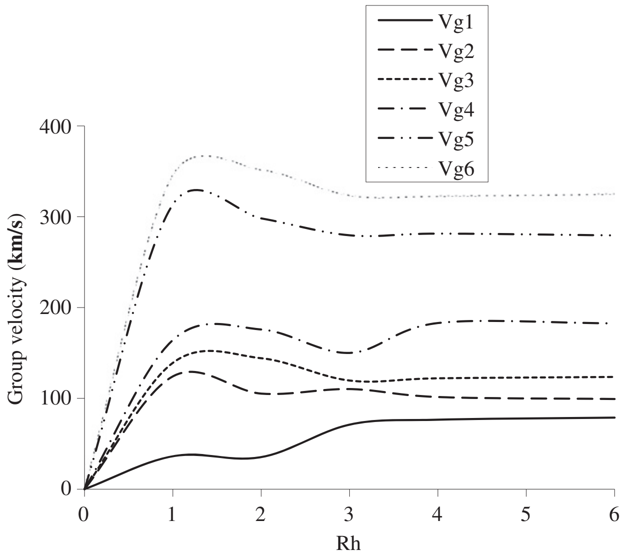

Fig. 4. Group velocity of extensional modes versus wavenumber (*Rh*) for CT theory.

158 km/s towards the thermoelastic Rayleigh wave velocity with increasing wavenumber as observed from Fig. 3, whereas in elastic plate it becomes close to Rayleigh wave velocity as can be seen from Figs.1–2. The phase velocity of higher extensional modes attain quite large values at vanishing wavenumber which sharply slash down to become steady and asymptotically close to shear wave velocity at extremely large wavenumbers. The group velocities of various modes are found to be zero at vanishing wavenumbers which correspond to the condition of zero energy transmission. The group velocity of various extensional modes of wave propagation both elastic (isothermal or isentropic) and thermoelastic plates increase monotonically in the interval $0 \le Rh \le 1$ and tend to phase velocity profiles of respective modes at high frequency limits. Group and phase velocity profiles coincide at higher values of the wavenumber (Rh) in both elastic and thermoelastic plates. The trends of variation of phase velocity of extensional modes in thermoelastic plate are found to be quite similar to that reported by Achenbach [22] and Graff [26] in elastokinetics except some modifications due to the thermomechanical coupling. According to Sharma et al. [23], at low frequencies mechanical energy transfer is more dominant than thermal conduction and hence at low frequency limits the wave-like modes are identified with the small amplitude waves in elastic material that does not transport heat. These may be regarded as inherent in the classical elastodynamics derived strictly from mechanical principles. Thus the behavior and character of all the existing possible modes of extensional motion of the thermoelastic and elastic plates are similar.

The phase and group velocities of flexural wave modes in elastic (isothermal/isentropic) and thermoelastic (CT) plates for different values of wavenumber (Rh) are plotted in Figs. 5–8. It is noticed that the magnitudes of phase and group velocities of various wave modes in coupled thermoelastic plate have small magnitudes as compared to that in elastic plate under isothermal or isentropic conditions at all wavelengths. The phase velocity (V_1) of fundamental mode of flexural motion increases from zero value towards the classical thermoelastic Rayleigh wave velocity with increasing wavenumber as observed from Fig. 7 in contrast to that in elastic plate where it becomes close to Rayleigh wave velocity as can be seen from Figs. 5 and 6. The phase velocities of higher flexural modes attain quite large values at vanishing wavenumber which sharply slash down to become steady and asymptotically closer to shear wave velocity at extremely large wavenumbers. The group velocities of various modes are found to be zero at vanishing wavenumber which corresponds to the condition of zero energy transmission. The group velocity of various flexural modes of wave propagation both elastic and thermoelastic plates increase monotonically in the interval $0 \le Rh \le 1$ and tend to phase velocity profiles of respective modes at high frequency limits. Figs. 9 and 10 show the variations of dimensional attenuation coefficients of extensional

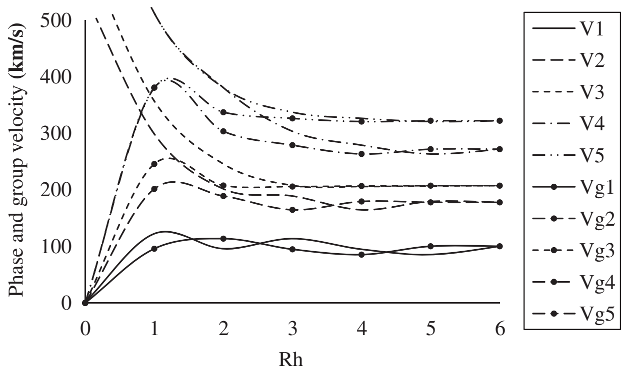

Fig. 5. Phase and group velocities of flexural modes in non-conducting (isentropic) elastic plate versus wavenumber (Rh).

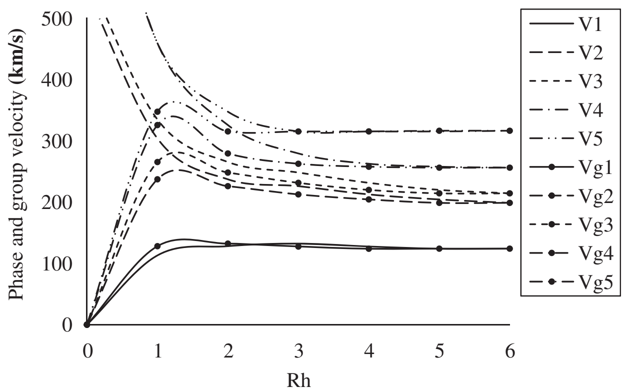

Fig. 6. Phase and group velocities of flexural modes in isothermal elastic plate versus wavenumber (Rh).

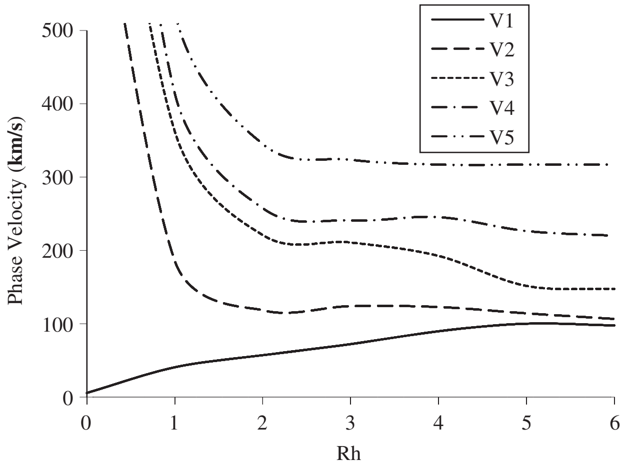

Fig. 7. Phase velocity of flexural modes versus wavenumber (Rh) for CT theory.

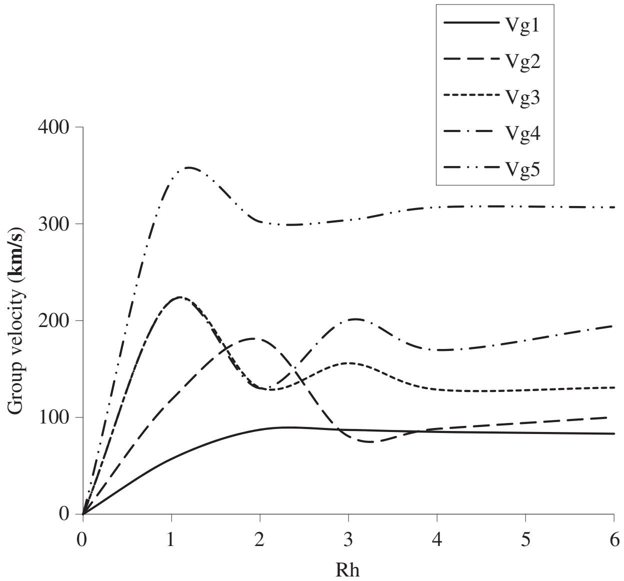

Fig. 8. Group velocity of flexural modes versus wavenumber (Rh) for CT theory.

and flexural motions versus wavenumber in the context of CT theory of thermoelasticity respectively. The trends of variations of phase velocity of flexural motion in thermoelastic plate are also found to be quite similar to that reported by Achenbach [22] and Graff [26] in elastokinetics except some modifications due to the thermomechanical coupling.

Tables 1–3 show the variations of phase velocities (V_i^n, $i=1,\ldots,6$), group velocities ($V_{g_i}^n$, $i=1,\ldots,6$) and attenuations coefficients (Q_i^n, $i=1,\ldots,6$) of extensional and flexural modes with respect to wavenumber (Rh). Here normalization of quantities has been done with their corresponding values in classical (CT) thermoelasticity so that $V_i^n = V_i^F/V_i^{CT}$, $V_{g_i}^n = V_{g_i}^F/V_{g_i}^{CT}$ and $Q_i^n = Q_i^F/Q_i^{CT}(i=1,\ldots,6)$, where F stands for LS or GL in case of respective non-classical theory of thermoelasticity. Tables 1 and 2 reveal that there are significant deviations in magnitudes of phase and group velocities of extensional and flexural wave modes in non-classical theories (LS, GL) from that of classical (CT) theory. It is observed from Table 1 that the magnitude of phase velocity (V_1^n) and group velocity ($V_{g_1}^n$) of fundamental mode of vibration is less than unity in low and high frequency regime. Thus the thermal relaxation time contributes in decreasing the magnitudes of phase and group velocities of extensional modes. The magnitudes of phase velocities (V_2^n, V_4^n, V_5^n, V_6^n) and group velocities ($V_{g_2}^n$, $V_{g_4}^n$, $V_{g_6}^n$)are greater than unity at low frequency regime and less than unity at high frequencies. However, the magnitudes of phase velocity (V_3^n) and group velocity ($V_{g_3}^n$) are less than unity at low frequencies and greater than unity at high frequency regime for both extensional and flexural motions of the plate in context of GL and LS theories of thermoelasticity.

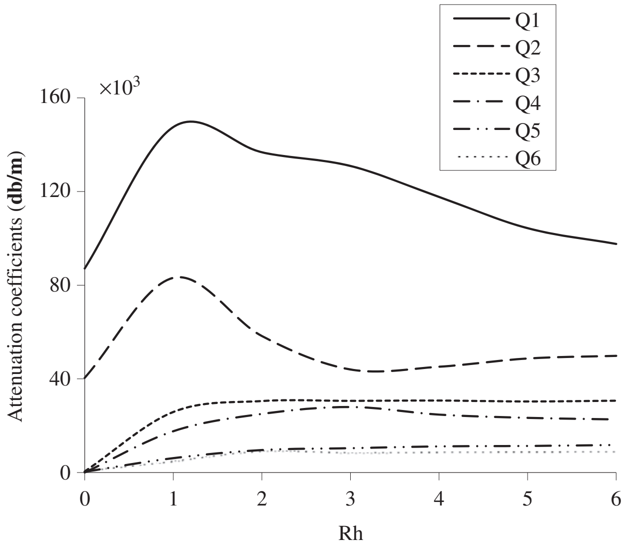

Fig. 9. Attenuation coefficients of extensional motion versus wavenumber (Rh) for CT theory.

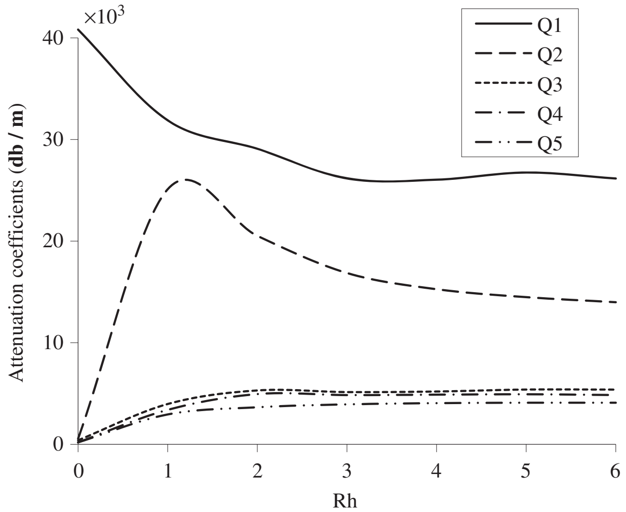

Fig. 10. Attenuation coefficients of flexural motion versus wavenumber (Rh) for CT theory.

In Table 2, the variations of phase velocity (V_i^n, $i=1,\ldots,5$) and group velocity ($V_{g_i}^n$, $i=1,\ldots,5$) of all the flexural modes versus wavenumber (Rh) in the context of generalized theories (GL, LS) of thermoelasticity have been shown. It is observed that the magnitudes of phase velocities (V_1^n, V_3^n, V_4^n, V_5^n)and group velocities ($V_{g_1}^n$, $V_{g_3}^n$, $V_{g_4}^n$, $V_{g_5}^n$)are greater than unity. Thus, the thermal relaxation time contributes in increasing the magnitudes of phase and group velocities of various modes of vibrations. The magnitudes of phase velocity (V_2^n) and group velocity($V_{g_2}^n$) are noticed to be less than unity at all the wavelengths for flexural motion except at certain wavelengths. It can be concluded that thermal relaxation times result in decrease of phase and group velocities of extensional wave modes in contrast to flexural one in which case these quantities increase with relaxation times.

The attenuation coefficients (Q_i^n, $i=1,\ldots,6$) of all the wave modes of extensional and flexural motion versus wavenumber (Rh) are shown in Table 3. It is noticed that the magnitude of attenuation coefficient (Q_1^n) of fundamental mode is less than unity for extensional motion and it is greater than unity in case of flexural motion at all the wavelengths. The attenuation coefficients (Q_2^n, Q_6^n)of extensional modes have magnitudes greater than unity at low frequencies but less than unity at high frequency regime for both GL and LS theories of thermoelasticity. The attenuation coefficient (Q_2^n)of flexural motion has magnitude less than unity at low and high frequency regime except at certain wavelengths. The magnitude of attenuation coefficient (Q_3^n)of extensional mode is less than unity and that of flexural mode is greater than unity at low frequency regime and vice-versa at high frequencies. The magnitudes of attenuation coefficient(Q_4^n) of extensional and flexural modes are less than unity at low frequency regime and greater than unity at high frequencies

Table 1
Phase (V_i^n) and group ($V_{g_i}^n$) velocities of extensional modes versus wavenumber (Rh).

Rh	V_1^n	$V_{g_1}^n$	V_2^n	$V_{g_2}^n$	V_3^n	$V_{g_3}^n$	V_4^n	$V_{g_4}^n$	V_5^n	$V_{g_5}^n$	V_6^n	$V_{g_6}^n$
GL												
0	0.9953	0	1.0041	0	0.9961	0	1.0000	0	1.0263	0	1.0000	0
1	0.9952	0.9955	1.0040	1.0048	0.9958	0.9955	0.9980	1.0000	1.0000	0.8879	1.0061	1.1268
2	0.9955	0.9942	1.0048	0.9844	0.9955	1.0065	1.0000	1.0086	0.8879	1.0856	1.1268	0.8957
3	0.9956	0.9938	0.9950	0.9958	1.0009	1.0008	1.0047	1.0024	0.9941	0.9946	1.0045	1.0105
4	0.9954	0.9981	0.9953	0.9960	1.0009	1.0008	1.0041	1.0048	0.9943	0.9874	1.0066	1.0063
5	0.9958	0.9959	0.9955	0.9929	1.0008	1.0008	1.0043	1.0047	0.9925	1.0018	1.0065	1.0063
6	0.9958	0.9958	0.9949	0.9949	1.0008	1.0008	1.0043	1.0043	0.9944	0.9944	1.0065	1.0065
LS												
0	0.9958	0	1.0041	0	0.9968	0	1.0000	0	1.0263	0	0.9737	0
1	0.9954	0.9955	1.0040	1.0048	0.9958	0.9955	1.0000	1.0000	1.0000	0.8898	1.0030	1.1241
2	0.9955	0.9965	1.0048	0.9861	0.9955	1.0065	1.0000	1.0086	0.8898	1.0876	1.1241	1.0243
3	0.9954	0.9955	0.9958	0.9965	1.0009	1.0008	1.0047	1.0024	0.9961	0.9964	1.0747	0.8903
4	0.9954	0.9980	0.9961	0.9967	1.0009	0.9976	1.0041	1.0048	0.9962	0.9964	1.0022	1.0105
5	0.9958	0.9930	0.9962	0.9935	1.0000	1.0000	1.0043	1.0048	0.9962	1.0054	1.0043	1.0042
6	0.9954	0.9954	0.9956	0.9956	1.0000	1.0000	1.0043	1.0043	0.9981	0.9981	1.0043	1.0043

Table 2
Phase (V_i^n) and group ($V_{g_i}^n$) velocities of flexural modes versus wavenumber (Rh).

Rh	V_1^n	$V_{g_1}^n$	V_2^n	$V_{g_2}^n$	V_3^n	$V_{g_3}^n$	V_4^n	$V_{g_4}^n$	V_5^n	$V_{g_5}^n$
GL										
0	0.9956	0	1.0002	0	0.1000	0	1.0000	0	0.9961	0
1	1.0026	1.0022	0.9891	0.9889	1.0000	1.0025	1.0112	1.0000	0.9808	0.0105
2	1.0022	1.0038	0.9889	0.9871	1.0024	0.9955	1.0000	1.0000	1.0105	1.0107
3	1.0028	0.9966	0.9882	0.9977	0.9981	1.0060	1.0000	1.0000	1.0106	1.0106
4	1.0017	0.9990	0.9914	0.9982	1.0025	1.0084	1.0000	0.9953	1.0106	1.0555
5	1.0013	1.0069	0.9934	0.9978	1.0049	0.9957	0.9985	0.9987	1.0215	1.0215
6	1.0025	1.0025	0.9945	0.9945	1.0028	1.0028	0.9986	0.9986	1.0215	1.0215
LS										
0	0.9955	0	1.0002	0	1.0000	0	1.0000	0	0.9964	0
1	1.0019	1.0026	0.9887	0.9882	1.0000	1.0000	1.0112	1.0000	0.9808	1.0000
2	1.0026	1.0032	0.9882	0.9861	1.0000	0.9970	1.0000	1.0000	1.0000	1.0000
3	1.0028	1.0000	0.9874	0.9969	0.9981	1.0060	1.0000	0.9970	1.0000	1.0000
4	1.0023	1.0000	0.9906	1.0000	1.0025	1.0084	0.9984	0.9988	1.0000	1.0000
5	1.0019	1.0075	0.9934	0.9978	1.0049	0.9957	0.9985	0.9987	1.0000	1.0555
6	1.0031	1.0031	0.9945	0.9945	1.0028	1.0028	0.9986	0.9986	1.0106	1.0106

except at certain wavelengths. The magnitudes of attenuation coefficient (Q_5^n)of extensional and flexural modes are magnitude greater than unity at all the wavelengths with same except at certain wavelengths.

10. Conclusions

The asymptotic operator plate model for free vibrations; both extensional and flexural, in a homogenous thermoelastic plate leads to sixth and fifth degree secular equations, respectively, that governs frequency and phase velocity of various possible modes of wave propagation at all wavelengths. Numeric computation to find phase velocity, group velocity and attenuation coefficient have been done by using MATHCAD software by using Graffe's root squaring method solve polynomial equations with complex coefficients. The infinite power series expansions of classical thermoelastic Rayleigh–Lamb frequency equation and secular equations obtained with operator plate model are found to be in close agreement up to the approximations of order $o(\eta^{10})$. It is observed that the non-trivial roots are almost same in case of both the techniques. Phase velocity of fundamental extensional and flexural wave modes in thermoelastic plate approach to thermoelastic Rayleigh wave and classical Rayleigh wave velocity at large wavenumbers, respectively. It is noticed that the thermal relaxation times have significant effect on low-frequency waves in the limiting case which supports the conclusion that the "second sound" effects are short lived. The group velocity profiles of all the wave modes approach to the phase velocity profiles of respective modes at short wavelengths for both extensional and flexural motions of the plate. Moreover, the phase and group velocities have same magnitudes in case of non-dispersive wave modes. Higher waves modes are more attenuated than the fundamental and second modes of wave propagation. It is also observed that the flexural wave

Table 3
Attenuation coefficients (Q_i^n) of extensional (E) and flexural (F) modes versus wavenumber (Rh).

Rh	Q_1^n		Q_2^n		Q_3^n		Q_4^n		Q_5^n		Q_6^n
	E	F	E	F	E	F	E	F	E	F	E
GL											
0	0.9958	0.9960	1.0042	0.9048	0.9948	1.0119	1.0845	1.0166	1.3826	1.0000	1.0000
1	0.9953	1.0232	1.0039	0.9970	0.9962	1.0102	0.9889	1.0000	1.0000	0.9949	1.0000
2	0.9955	1.0176	1.0049	0.9971	0.9958	1.0076	0.9400	0.9973	1.1862	1.0111	0.8416
3	0.9955	1.0168	0.9954	0.9977	1.2000	1.0157	1.0051	0.9944	1.0117	1.0103	0.9902
4	0.9959	1.0135	1.0340	1.0011	1.0070	0.9926	1.0045	1.0104	1.0072	1.0390	0.9905
5	0.9957	1.0072	0.9950	0.9972	1.0517	1.0000	1.0047	1.0182	0.6101	1.0000	0.9906
6	0.9954	1.0042	0.9943	0.9971	1.0606	1.0075	1.0047	1.0476	1.0069	1.0099	0.9861
LS											
0	0.9958	0.9960	1.0042	0.8965	0.9961	1.0119	1.0806	1.0047	1.3842	1.0042	1.0000
1	0.9956	1.0025	1.0039	0.9970	0.9962	1.0102	0.9889	0.8474	1.0000	0.9949	1.0000
2	0.9955	1.0197	1.0042	0.9971	0.9958	1.0076	0.9800	0.9973	1.1808	1.0111	0.8461
3	0.9955	1.0179	0.9954	0.9973	1.1333	1.0157	1.0042	0.9962	1.0078	1.0000	1.2683
4	0.9955	1.0145	1.0349	1.0005	1.0465	0.9926	1.0045	1.0104	1.0036	1.0000	0.9953
5	0.9953	1.0080	1.0446	0.9966	1.0172	1.0000	1.0047	1.0182	0.6101	1.0000	0.9953
6	0.9954	1.0056	0.9951	0.9965	1.0303	1.0075	1.0047	1.0000	1.0069	1.0000	0.9954

modes are less attenuated than extensional one. The thermal variations result in reduction of phase and group velocities of the wave modes because of their attenuating character. The asymptotic differential equations which govern flexural and extensional motions can be written from the respective frequency equations obtained here without any difficulty as was done by Losin [15,16] though order of equations will be higher here. Operator plate model approximates thin and thick plates structures more accurately than the other methods. Because it allows elimination of restrictions due to the convergence interval for the infinite matrix series and permits applicability of the model for short and long wavelength limits. Moreover, the derived dispersion relations give good approximations without any correction factors as in the Reissener–Mindlin theory.

Acknowledgment

The valuable suggestions of the reviewers for the improvement of this work are thankfully acknowledged.

Appendix

The quantities a_i, b_i, d_i (i=1, 2, 3) and q_i (i=1, 2, 3, 4, 5) used in Eq. (21) are defined as

$$a_1 = 4(\delta^4 - 2 + \varepsilon_1\delta_1\delta^2\tau_1^{-1}),$$
$$a_2 = 4\delta^2[\delta^2(\delta^2+5) - 10 + 2\tau_1^{-1}\varepsilon_1\{7+\delta^2(1-2\delta^2-\delta_1\tau_0)\}] - 16 \tag{A.1}$$

$$a_3 = 1 + 5\delta^4(2\delta^{-2} + 1 - \tau_1^{-1}\varepsilon_1), \quad b_1 = 2\tau_1^{-1}\varepsilon_1\delta\delta_1(\tau_0\delta^2 - \overline{\delta}_1),$$
$$b_2 = -10\tau_1^{-1}\varepsilon_1\delta\delta_1(\tau_0\delta^2 + \overline{\delta}_1) \tag{A.2}$$

$$b_3 = -2\tau_1^{-1}\varepsilon_1\delta[1+\delta^6\{\delta_1\delta^{-4}(\tau_0{}^2+5)+\tau_0-1\}], \quad d_1 = 3+2\delta^2(1-\tau_0-\varepsilon_1) \tag{A.3}$$

$$d_2 = 4\delta^2[1-\tau_0+5(\delta^2\tau_0-2)], \quad d_3 = 10+\delta^2[5(1-\tau_0)+2\delta^2\{1+\tau_0(\tau_0-6)\}] \tag{A.4}$$

$$q_1 = \delta^2[3+\tau_0(1-6\delta^4) - \tau_1^{-1}\varepsilon_1], \quad q_2 = -\delta^4[\tau_0(3-\tau_0)+\tau_1^{-1}\varepsilon_1(1-2\tau_0)] \tag{A.5}$$

$$q_3 = 2[\delta^2\{\tau_1^{-1}\varepsilon_1(3-5\delta^2)+2\tau_0(8-15\delta^2)+5(2+\delta^2)\}-5] \tag{A.6}$$

$$q_4 = \delta^2[\tau_0\{\delta^2(37-10\delta^2)-10\}+10\tau_1^{-1}\varepsilon_1\{1+\tau_0\delta^2(1-2\delta^2)\}-5\delta^2] \tag{A.7}$$

$$q_5 = \delta^6[\tau_0(5-10\tau_0+\tau_0{}^2)+\tau_1^{-1}\varepsilon_1\{1+3\tau_0(\tau_0-4)\}] \tag{A.8}$$

The quantities B_i (i=1 to 7) and p_i (i=1 to 10) used in Eq. (26) are given as

$$B_1 = m_6(p_2+p_3)+m_7(p_2+\gamma_1)+p_3(4\delta^2-3)]+240\delta^{-2}[m_4p_5-m_5p_4-10\delta\tau_1(2p_7-m_3p_6) \tag{A.9}$$

$$B_2 = 20\{p_2\delta^2 + p_3(3\delta^2-1) - 4m_5(5\delta^2-1)\} - m_6 p_1 - m_7(p_1-\beta_2) \tag{A.10}$$

$$B_3 = m_7\gamma_1 + (4\delta^2-3)(p_2+p_3)] + 240\delta^{-2}[p_5(1+m_4) + (m_5 - m_4 p_4) - 10\delta\tau_1(2p_7 - m_3 p_6)] \tag{A.11}$$

$$B_4 = m_6(p_1-\gamma_2) - m_7\gamma_2 + (4\delta^2-3)p_1 + 20\{(\gamma_1 + 3m_2 + 96m_5 - 120m_4)\delta^2 + p_3\gamma_6 - m_2 + 24 \tag{A.12}$$

$$B_5 = \delta^2\left[\frac{1}{10}p_1(1+3\delta^2) + \delta^2\left\{\frac{1}{10} + p_2 m_5 + 240\gamma_4 - 144\varepsilon_1\delta^2(2-\delta^2)\right\} + p_3 - 48\gamma_4\right] \tag{A.13}$$

$$B_6 = \delta^2[3(p_2+\gamma_1) - \gamma_1 m_6 - 4\delta^2(p_2+\gamma_1)] - 24[p_4 - m_4\gamma_7 + 2p_0\delta\tau_1] \tag{A.14}$$

$$B_7 = 24\delta^2(4m_4-5) + \gamma_1(3\delta^2-1) + p_2\gamma_6 + 24 + 0.05\{m_6\gamma_2 - (4\delta^2-3)(p_1-\gamma_2)\} \tag{A.15}$$

$$\begin{gathered} p_0 = 2\delta^3\varepsilon_1(1-5\delta^2+2\delta^4), \quad p_1 = 288[-1+\delta^2(\tau_0+\varepsilon_1)], \\ p_2 = 24[6-m_1+2\delta^2\{(7\delta^2-10)\tau_0+\varepsilon_1\}] \end{gathered} \tag{A.16}$$

$$\begin{gathered} p_3 = 24[1+\delta^2\tau_0(\tau_0\delta^2-6) - \varepsilon_1(1-6\delta^{-2}+2\tau_0)], \\ p_4 = 4\delta^2[16+\delta^2\{13\delta^2-37+\varepsilon_1(1+2\delta^2)\}] \end{gathered} \tag{A.17}$$

$$\begin{gathered} p_5 = \delta^2[23+\delta^4\{11-50\delta^{-2}+\varepsilon_1(2+9\delta^4)\}], \\ p_6 = \varepsilon_1\tau_1^{-1}\delta^3[2+\delta^4\{(5+\tau_0)-14\delta^{-2}] \end{gathered} \tag{A.18}$$

$$\begin{gathered} p_7 = \varepsilon_1\tau_1^{-1}\delta^7(1+\tau_0-4\delta^{-2}), \quad p_8 = 2\delta^5\tau_1(14\delta^{-2}+\tau_0+\varepsilon_1-31), \\ p_8 = 2\delta^5\tau_1(14\delta^{-2}+\tau_0+\varepsilon_1-31) \end{gathered} \tag{A.19}$$

$$\begin{gathered} p_{10} = -8\delta^3\tau_1(5\delta^2-2), \quad \gamma_1 = 192(3-2\delta^2), \\ \gamma_2 = 576(2-\delta^2), \quad \gamma_3 = -\varepsilon_1\tau_1^{-1}\delta(2-\delta^2) \end{gathered} \tag{A.20}$$

$$\gamma_4 = \delta^2(\tau_0+\varepsilon_1), \quad \gamma_5 = 3-2\delta^2, \quad \gamma_6 = 2\delta^2-1, \quad \gamma_7 = 8\delta^2[5(1+\delta^4)-12\delta^2] \tag{A.21}$$

The quantities r_i (i=1 to 10) and $\overline{\varphi}_i$ (i=2, 4) utilized in Eq. (34) are defined as

$$r_1 = -10\delta^2 + \overline{\varphi}_2(19-24\delta^2+20\overline{\varphi}_2) + \varphi_4[31-12(2\delta^2+\overline{\varphi}_2)] + \varphi_2[10-43\delta^2+12(\varphi_2+2\varphi_4)] \tag{A.22}$$

$$r_2 = 5-68\delta^2 + 4\overline{\varphi}_2(4-6\delta^2) + 8\varphi_4(8-3\delta^2) + 4\varphi_2(17-28\delta^2-16\overline{\varphi}_2+12\varphi_2-9\varphi_4) \tag{A.23}$$

$$r_3 = -10\delta^2 + \overline{\varphi}_2(19-24\delta^2+20\overline{\varphi}_2) + \varphi_4[31-12(2\delta^2+\overline{\varphi}_2)] + \varphi_2[10-43\delta^2+12(\varphi_2+2\varphi_4)] \tag{A.24}$$

$$r_4 = -41\delta^2 + 14 + 4\overline{\varphi}_2\delta_1 + 4\varphi_4(\delta_1+4) + \varphi_2[25-4\delta^2+16(\varphi_2+\delta_1-2\overline{\varphi}_2)] \tag{A.25}$$

$$r_5 = 15\delta^2 - 1 + 8\overline{\varphi}_4 - 2\varphi_4(4\delta_1+5) + \varphi_2[13\delta^2-23\delta_1+2\varphi_2(4\overline{\varphi}_2-5)-8\varphi_4] \tag{A.26}$$

$$r_6 = -2\delta^2 + 54\overline{\varphi}_2\delta_1 + \varphi_4(2+5\delta_1) + \varphi_2(2-6\delta^2+2\varphi_2+5\varphi_4-4\overline{\varphi}_4) \tag{A.27}$$

$$r_7 = 39\delta^2 - 6 + 4\overline{\varphi}_2(\delta^2+3) + 4\varphi_4(\delta^2-3) - \varphi_2(39-36\delta^2+24\varphi_2+2\varphi_4+36\overline{\varphi}_2) \tag{A.28}$$

$$r_8 = 18\delta^2 - 1 + \overline{\varphi}_2(12\delta^2+49) + \varphi_4(12\delta^2-1) - 3\varphi_2(6-13\delta^2+12\varphi_2+4\varphi_4) \tag{A.29}$$

$$r_9 = 3\delta^2 + \overline{\varphi}_2(6+13\delta^2+12\overline{\varphi}_2\varphi_2) + \varphi_4(13\delta^2-6) - \varphi_2(3-18\delta^2+12\varphi_2+13\varphi_4) \tag{A.30}$$

$$r_{10} = \overline{\varphi}_2(1+6\delta^2+\overline{\varphi}_4) + \varphi_4(6\delta^2-1) + \varphi_2(3\delta^2-2\varphi_2-6\varphi_4) \tag{A.31}$$

$$\varphi_i = \delta^i(a^i+b^i), \quad \overline{\varphi}_i = \delta^{2i}a^i b^i, \quad i = 2, 4 \tag{A.32}$$

References

[1] W. Nowacki, *Dynamic Problems of Thermoelasticity*, *Noordhoff*, 1975.
[2] H.W. Lord, Y. Shulman, The generalized dynamical theory of thermoelasticity, *Journal of Mechanics and Physics of Solids* 15 (1967) 299–309.
[3] A.E. Green, K.A. Lindsay, Thermoelasticity, *Journal of Elasticity* 2 (1972) 1–7.
[4] D.S. Chandrasekharaiah, Thermoelasticity with second sound—a review, *Applied Mechanics Reviews* 39 (1986) 355–376.
[5] C.C. Ackerman, B. Bentman, H.A. Fairbank, R.A. Krumhansal, Second sound in helium, *Physical Review Letters* 16 (1966) 789–791.

[6] R.A. Guyer, J.A. Krumhansal, Thermal conductivity, second sound and phonon hydrodynamic phenomenon in nonmetallic crystals, *Physical Review Letters* 148 (1966) 778–781.
[7] C.C. Ackerman, W.C. Overtone, Second sound in helium-3, *Physical Review Letters* 22 (1969) 764–766.
[8] J.N. Sharma, D. Singh, R. Kumar, Generalized thermoelastic waves in homogeneous isotropic thermoelastic plate, *Journal of Acoustical Society of America* 108 (2000) 848–851.
[9] J.N. Sharma, On the propagation of thermoelastic waves in homogeneous isotropic plate, *Indian Journal of Pure and Applied Mathematics* 32 (2001) 1329–1341.
[10] R. Kirova, V. Georgiev, B. Rubino, R. Sampalmieri, B. Yordanov, Asymptotic behavior for linear and nonlinear elastic waves in materials with memory, *Journal of Non-Crystalline Solids* 354 (2008) 4126–4137.
[11] N.G. Ryabenkov, R.F. Faizullina, The common asymptotic nature of methods of solving problems of the theory of elasticity for slabs and plates, *Journal of Applied Mathematics and Mechanics* 70 (2006) 399–407.
[12] L.A. Agalovyan, R.S. Gevorkyan, Asymptotic solution of the first boundary-value problem of the theory of elasticity of the forced vibrations of an isotropic strip, *Journal of Applied Mathematics and Mechanics* 72 (2008) 452–460.
[13] C. Gales, On the asymptotic spatial behavior in the theory of mixtures of thermoelastic solids, *International Journal of Solids and Structures* 45 (2008) 2117–2127.
[14] R.S. Gevorgyan, Asymptotic solution of coupled dynamic problems of thermoelasticity for isotropic plates, *Journal of Applied Mathematics and Mechanics* 72 (2008) 87–91.
[15] N.A. Losin, Asymptotics of flexural waves in isotropic elastic plates, ASME, *Journal of Applied Mechanics* 64 (1997) 336–342.
[16] N.A. Losin, Asymptotics of extensional waves in isotropic elastic plates, ASME, *Journal of Applied Mechanics* 65 (1998) 1042–1047.
[17] N.A. Losin, On the equivalence of dispersion relations resulting from Rayleigh-Lamb frequency equation and the operator plate model, *Journal of Vibration and Acoustics* 123 (2001) 417–420.
[18] J.N. Sharma, P.K. Sharma, S.K. Rana, Flexural and transversal wave motion in homogeneous isotropic thermoelastic plates by using asymptotic method, *Journal of Sound and Vibration* 329 (2010) 804–818.
[19] A.M. Protsenko, Asymptotics of wave problems for a cylindrical shell, (Izvesic ANSSSR), *Applied Mathematics and Mechanics* 44 (1980) 507–515.
[20] D.V. Strunin, On characteristic times in generalized thermoelasticity, *Journal of Applied Mathematics* 68 (2001) 816–817.
[21] A.E. Green, A note on linear thermoelasticity, *Mathematica* 19 (1972) 69–75.
[22] J. D. Achenbach, Wave Propagation in Elastic Solids, North-Holland Publishing Company- Amsterdam, London 1973.
[23] J.N. Sharma, V. Kumar, S.P. Sud, Plane harmonic waves in orthorhombic thermoelastic materials, *Journal of Acoustics Society America* 107 (2000) 293–305.
[24] J.N. Sharma, R. Kumar, Asymptotics of wave motion in transversely isotropic plates, *Journal of Sound and Vibration* 274 (2004) 747–759.
[25] J.N. Sharma, *Numerical Method for Engineers and Scientists*, Second edition, Narosa Publishing House Pvt. Ltd., New Delhi, 2007.
[26] K.F. Graff, *Wave Motion in Elastic Solids*, Dover Publication, Inc., New York, 1975.